

# Critical Behavior in Topological Ensembles

K. Bulycheva<sup>1,5</sup>, A. Gorsky<sup>1,2</sup>, and S. Nechaev<sup>3,4</sup>

<sup>1</sup>*Institute of Information Transition Problems, B.Karetnyi 19, Moscow, Russia*

<sup>2</sup>*Moscow Institute of Physics and Technology, Dolgoprudny 141700, Russia*

<sup>3</sup>*Université Paris-Sud/CNRS, LPTMS, UMR8626, Bât. 100, 91405 Orsay, France*

<sup>4</sup>*P.N.Lebedev Physical Institute, RAS, 119991 Moscow, Russia*

<sup>5</sup>*Department of Physics, Princeton University, USA*

We consider the relation between three physical problems: 2D directed lattice random walks, ensembles of  $T_{n,n+1}$  torus knots, and instanton ensembles in 5D SQED with one compact dimension in  $\Omega$  background and with 5D Chern-Simons term at the level one. All these ensembles exhibit the critical behavior typical for the "area+length+corners" statistics of grand ensembles of 2D directed paths. Using the combinatorial description, we obtain an explicit expression of the generating function for  $q$ -Narayana numbers which amounts to the new critical behavior in the ensemble of  $T_{n,n+1}$  torus knots and in the ensemble of instantons in 5D SQED. Depending on the number of the nontrivial fugacities, we get either the critical point, or cascade of critical lines and critical surfaces. In the 5D gauge theory the phase transition is of the 3rd order, while in the ensemble of paths and ensemble of knots it is typically of the 1st order. We also discuss the relation with the integrable models.

## I. INTRODUCTION

Challenging questions appear often at edges of traditional fields. As an example, the new branch of mathematical physics, the "statistical topology" emerged recently by absorbing ideas from the statistical physics, theory of integrable systems, and algebraic topology (see, [1] for review). The scope of the statistical topology includes, on the one hand, mathematical problems involved in the construction of topological invariants of knots and links based on solvable models and, on the other hand, the physical and statistical problems related to summation over knot ensembles. In this work, we dwell predominantly to problems of the latter kind, demonstrating the emergence of a critical behavior in ensemble of  $T_{n,n+1}$  torus knots. This critical behavior is formulated in terms of knot invariants.

Torus knots  $T_{m,n}$  seem to be among the simplest objects in the knot theory. It is difficult to overestimate their role in different branches of mathematical physics. The topology of a torus knot is uniquely determined by the pair  $(m, n)$ , which fixes windings along two torus periods. In the Fig.1 few particular examples of torus knots,  $T_{2,3}$ ,  $T_{5,6}$ ,  $T_{10,11}$  from the series  $T_{n,n+1}$  are depicted. The closed curves wrap around the torus which is not shown.

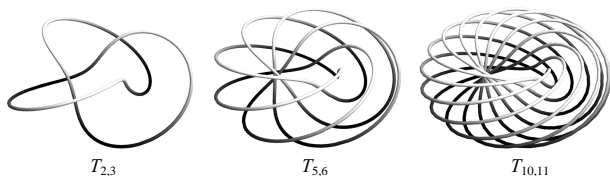


Figure 1: Few samples of torus knots from the series  $T_{n,n+1}$ :  $T_{2,3}$ ,  $T_{5,6}$ ,  $T_{10,11}$ .

Various explicit expressions for knot invariants are

known, ranging from classical Jones–Kauffman polynomials [2, 3], to recent superpolynomials of torus knots [4]. New approaches to the construction of torus knot invariants for particular knots, based on the application of topological string theories and deformed matrix models, have been formulated relatively recently in [5–7]. Much less is known about properties of *knot ensembles*, where the particular topology of a knot diagram is considered as a topologically "quenched" variable similar to the quenched disorder in statistical physics. The weighted summation over different torus knot types (i.e. different pairs  $(m, n)$ ) means the consideration of the grand canonical ensemble (i.e. of the generating function) of torus knots.

In this work we uncover the relation between three physical problems: i) two-dimensional directed (i.e. (1+1)-dimensional) lattice random walks with fixed area under the curve, ii) ensemble of  $T_{n,n+1}$  torus knots, and iii) a five-dimensional SQED with the Chern–Simons term at the level one. The reason for random walks to appear in ii) and iii) can be intuitively explained as follows. The main tool for the evaluation of the torus knot superpolynomials [4] and of Nekrasov partition function in the SUSY gauge theory [19], is the "equivariant localization" approach, which reduces the integral over the particular moduli space to the summation over the Young tableau. The last problem can be reformulated as a weighted sums over directed paths on a square lattice. The schematic relations between the problems considered in the paper is shown in the flowchart in the Fig.2.

Our main goal is the investigation of the critical behavior in these topological ensembles. From the "random walk side" the critical behavior occurs in the space of fugacities and we shall focus at the "area+length+corners" statistics of paths. We derive an explicit expression for the free energy in the random walks problem for the "area+length+corners" statistics, which provides the

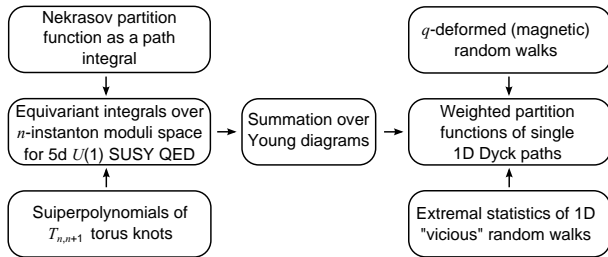


Figure 2: Schematic connection between different systems discussed in at length of the paper.

very nontrivial phase structure in the three-dimensional space of fugacities. In the generic case we find a cascade of phase transitions and in the degenerate cases we reproduce the previously known results.

The relation between the random walks and the torus knot superpolynomials can be traced from the mathematical literature, however the relation with the particular observable in 5D SQED instanton ensemble is new. Collecting different points of view, and the results of [35], we interpret the generating function of ensemble of  $T_{n,n+1}$  knots as the weighted sum of instanton contributions to the particular observable in 5D SQED and analyze the structure of corresponding generating function. The interpretation of the random walk fugacities in terms of the generating parameters in the torus knot ensemble is quite straightforward, moreover, these fugacities are identified in the 5D SQED as well: the corresponding parameters turn out to be the gauge coupling, the mass of the hypermultiplet, and the parameters of the  $\Omega$ -deformation.

Given an explicit expression for the free energy in the random walks problem for the "area+length+corners" statistics, we use it to analyze the ensembles of knots and instantons. We show that at the "gauge theory side" the particular third derivative of the instanton partition function with respect to the masses exhibits an unexpected critical behavior at some critical line in the space of parameters. We provide a physical interpretation of critical behaviors in all three theories considered here. Having an exact expression for the  $q$ -Narayana numbers (the generating function of the area- and corner-weighted (1+1)D Brownian excursion), we describe explicitly the phase portrait of different ensembles. We show that there is a 3rd order phase transition in the instanton ensemble, corresponding to the 1st order phase transition in the ensembles of random paths and torus knots.

The paper is organized as follows. In the Section 2 we describe the sum over the paths with different statistics and focus at the "area+length+corners" ensembles. In the Section 3 we discuss the representation of the partition function of (1+1)D Brownian excursions (Dyck paths) for the superpolynomials of  $T_{n,n+1}$  torus knots. The Section 4 is devoted to the identification of the sum over the paths as the instanton contribution to the particular observable in the 5D SQED. In the Section 5 we

consider the physical interpretation of the critical behavior in the space of fugacities in all models discussed at length of the paper. The relation with the Toda-like integrable system is mentioned in Section 6. Our findings and the open questions are summarized in the Conclusion. In the Appendix A, using the combinatorial description, we derive the new explicit expression for the generating function of  $(q, a, s)$ -Narayana polynomials (the generating function of area- and corner-weighted Dyck paths). In the Appendix B we remind for completeness the relation of the critical behavior with the "hydrodynamic description" of the edge singularities in GUE matrix ensembles.

## II. CRITICAL BEHAVIOR OF AREA- AND CORNER-WEIGHTED DYCK PATHS

By definition, the Dyck path of length  $2n$  on the square lattice starts at the origin  $(0,0)$ , ends at point  $(n,n)$  and consists of the union of sequential elementary " $\uparrow$ " and " $\rightarrow$ " steps, such that the path always stays above the diagonal of the square – see the Fig.3. The number of all Dyck paths of length  $2n$  is given by the Catalan number,  $C_n = \frac{1}{n+1} \binom{2n}{n}$ . We also denote Dyck paths as "Brownian excursions" (BE), having in mind an image of a charged particle on a square lattice in an external transversal magnetic field (after applied Wick rotation), where the motion of a particle is subject to two restrictions: it moves only up and right and never intersects the diagonal. Calculating the action for such a particle, we see that  $q = \exp(i\text{external magnetic field})$  is the fugacity of the area,  $A$ , under the Dyck path, and the  $s = \exp(\text{mass})$  is the fugacity of the path length,  $n$ . The information about statistics of area-weighted Dyck paths can be easily extracted from the generating function, which is the sum over all path lengths. This model can be referred to as the "chiral Hofstadter system", considered in [13].

To proceed, turn the lattice by  $\pi/4$  and write the recursion relation for the partition function  $Z_k(x; q)$  on a half-line,  $x \geq 0$ , where  $x$  is the height of the path at the step  $k$ . The area,  $A$ , below the path is counted as a sum of filled plaquettes (i.e. "heights") between the path and the  $x = 0$ -axis, as shown in the Fig.4a. Each plaquette has the weight  $q$ .

The partition function of area-weighted Dyck paths,  $Z_k(x; q)$ , satisfies the relation

$$\begin{cases} Z_{k+1}(x; q) = q^{x-1} Z_k(x+1; q) + Z_k(x-1; q) \\ Z_k(0; q) = 0 \\ Z_{k=0}(x; q) = \delta_{x,1} \end{cases} \quad (1)$$

Equivalently, (1) can be written in a matrix form. Define

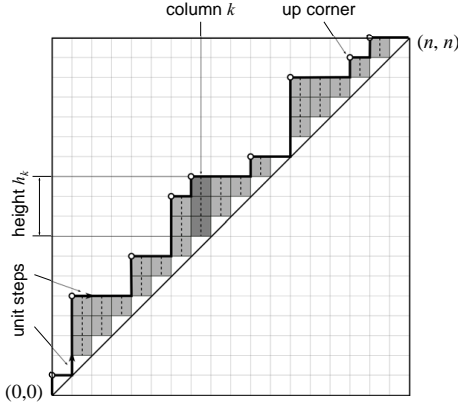


Figure 3: Sample of a Dyck path (Brownian excursion above the diagonal of the square) with a fixed area between the path and diagonal counted in full plaquettes (grey boxes) and a fixed number of up-corners (local "peaks" shown by open dots). The partition function of such paths is given by a generalization of  $q$ -Narayana polynomials.

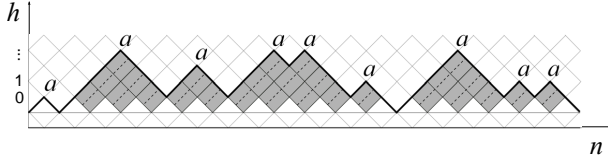


Figure 4: (a) The  $2n$ -step Dyck path (the same figure as in the Fig.3) turned by  $\pi/2$ , the heights are measured along the dotted lines and the  $\wedge$ -corners have fugacity  $a$ .

$\mathbf{Z}_n(q) = (Z_n(1; q), Z_n(2; q), Z_n(3; q), \dots)^T$ . Then

$$\mathbf{Z}_n(q) = T^n(q) \mathbf{Z}_0 \quad (2)$$

where

$$T(q) = \begin{pmatrix} 0 & 1 & 0 & 0 & \dots \\ 1 & 0 & q & 0 & \dots \\ 0 & 1 & 0 & q^2 & \dots \\ 0 & 0 & 1 & 0 & \dots \\ \vdots & \vdots & & & \ddots \end{pmatrix}; \quad \mathbf{Z}_0 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \end{pmatrix} \quad (3)$$

We are interested in the value  $Z_n(1; q)$  since at the very last step the trajectory returns to the initial point. Evaluating powers of the matrix  $T$ , we can straightforwardly check that the values of  $Z_N(1; q)$  are given by Carlitz-Riordan  $q$ -Catalan numbers, namely,

$$Z_n(1; q) = \begin{cases} C_{n/2}(q) & \text{for } n = 2k \ (k = 1, 2, \dots) \\ 0 & \text{for } n = 2k - 1 \ (k = 1, 2, \dots) \end{cases} \quad (4)$$

Recall that  $C_n(q)$  satisfy the recursion

$$C_n(q) = \sum_{k=0}^{n-1} q^k C_k(q) C_{n-k-1}(q) \quad (5)$$

which is the  $q$ -extension of the standard recursion for Catalan numbers. The generating function

$$F(s, q) = \sum_{n=0}^{\infty} s^n C_n(q)$$

obeys the functional relation

$$F(s, q) = 1 + sF(s, q)F(sq, q) \quad (6)$$

It is known that the solution of (6) can be written as a continued fraction expansion,

$$F(s, q) = \frac{1}{1 - \frac{s}{1 - \frac{sq}{1 - \frac{sq^2}{1 - \dots}}}} = \frac{A_q(s)}{A_q(s/q)} \quad (7)$$

where  $A_q(s)$  is the  $q$ -Airy function,

$$A_q(s) = \sum_{k=0}^{\infty} \frac{q^{k^2} (-s)^k}{(q; q)_k}; \quad (t; q)_k = \prod_{k=0}^{k-1} (1 - tq^k) \quad (8)$$

Let us describe how the critical behavior emerges at the Brownian excursion side. In the works [14–16] it has been shown that in the double scaling limit  $q \rightarrow 1^-$  and  $s \rightarrow \frac{1}{4}^-$  the function  $C(s, q)$  has the following asymptotic form

$$C(z) \sim C_{\text{reg}} + (1 - q)^{1/3} \frac{d}{dz} \ln \text{Ai}(4z); \quad z = \frac{\frac{1}{4} - s}{(1 - q)^{2/3}}, \quad (9)$$

where  $C_{\text{reg}}$  is the regular part at  $(q \rightarrow 1^-, s \rightarrow \frac{1}{4}^-)$  and  $\text{Ai}(z) = \frac{1}{\pi} \int_0^{\infty} \cos(\xi^3/3 + \xi z) d\xi$  is the Airy function.

The function  $C(s, 1)$  is the generating function for the undeformed Catalan numbers:

$$C(s, q = 1) = \frac{1 - \sqrt{1 - 4s}}{2s} \quad (10)$$

The generating function  $C(s, 1)$  is defined for  $0 < s < \frac{1}{4}$ , and at the point  $s = \frac{1}{4}$  the first derivative of  $C(s, 1)$  experiences a singularity which is interpreted as the critical behavior. The limit  $q = 1, s \rightarrow \frac{1}{4}^-$  can be read also from the asymptotic expression for  $C(s, q)$ , Eq.(9):

$$C(s, q)|_{q \rightarrow 1^-} \sim C_{\text{reg}} - 2\sqrt{1 - 4s}. \quad (11)$$

where  $C_{\text{reg}}$  is the regular part of  $C(s, q)$  at  $q \rightarrow 1$  and  $s \rightarrow \frac{1}{4}$  and  $C_{\text{reg}} = 2$  at  $s = \frac{1}{4}$ . Note that the first non-singular term in (11) does not contain  $q$ , so it is no matter in which order the limit in (9) is taken. However to define the double scaling behavior and derive the Airy-type asymptotic, the simultaneous scaling in  $s$  and  $q$  is required.

The generating function,  $F(a, s)$ , for Narayana numbers, which count Dyck paths with fixed fugacity of corners,  $a$ , demonstrates the behavior similar to (11), namely the square-root singularity,

$$F(a, s) = \frac{1 - (1 - a)s - \sqrt{(1 + s - sa)^2 - 4s}}{2s} \quad (12)$$

This behavior at  $a = 1$  coincides with the one of Catalans (11). In Appendix A we have derived the explicit expression for the generating function  $F(q, a, s)$  of  $q$ -Narayanas with the Airy-type asymptotic (9) in terms of  $q$ -orthogonal polynomials related to Rogers–Ramanujan continued fractions,

$$F(q, a, s) = \frac{A_q(s; s(1 - a))}{A_q(s/q; s(1 - a)/q)} \quad (13)$$

where  $A_q(s; s(1 - a))$  is the extension of the  $q$ -Airy function  $A_q(s)$  defined in (8). The function  $A_q(s; s(1 - a))$  reads

$$A_q(s; s(1 - a)) = \sum_{k=0}^{\infty} \frac{q^{k^2} (-s)^k}{(q; q)_k (-s(1 - a); q)_k} \quad (14)$$

One can immediately see that  $A_q(s, s(1 - a))|_{a=1} = A_q(s)$ , where  $A(s)$  is given by (8).

Completing this Section it is worth reminding that appearance of the singularity of type (9) is the manifestation of the third-order phase transition. In the seminal paper [33] it has been shown that the largest eigenvalue,  $\lambda_n$ , of the Gaussian  $n \times n$  random matrix ensemble, converges at  $n \rightarrow \infty$  to  $\lambda_n \rightarrow 2\sqrt{n} + n^{1/6}\chi$ , where the random variable  $\chi$  has a limiting  $n$ -independent distribution,  $\text{Prob}(\chi \leq x) = F_{\text{GUE}}(x)$ , being the so-called Tracy–Widom distribution for GUE ensemble [34]. So, the normalized value  $\Lambda_n = \lambda_n/\sqrt{n}$  at large (but finite)  $n$  has an uncertainty (i.e. the width of the distribution) of order of  $n^{-1/3}$ , typical for the 3rd order phase transitions. Above and below the critical value  $\Lambda_\infty = \lim_{n \rightarrow \infty} \Lambda_n = 2$ , the tails of the distribution  $P(\Lambda)$  have different asymptotics, signifying existence of strong (for  $\Lambda < \Lambda_\infty$ ) and weak (for  $\Lambda > \Lambda_\infty$ ) couplings.

### III. DYCK PATHS GENERATING FUNCTIONS AND TORUS KNOT INVARIANTS

In this Section we briefly explain the relation between the invariants of knots and the random walks with three fugacities. We restrict ourselves by the torus knots,  $T_{n,m}$ , and consider the superpolynomial introduced in [17]. They depend on three generating parameters, which can be related to the fugacities of 2D directed random walk. Remind that the superpolynomial is the Poincaré polynomial of the triply graded Khovanov homologies,  $H_{ijk}$ , which in the vector space can be attributed to the

knot. The knot superpolynomials are the generalizations of the HOMFLY knot polynomials and depend on three variables, corresponding to gradings [17]

$$P_{n,m}(a, q, t) = \sum_{ijk} a^i q^j t^k \dim H_{ijk} \quad (15)$$

At  $t = q^{-1}$  the superpolynomial  $P_{n,m}(a, q, t)$  reduces to the standard HOMFLY polynomial. Alternatively, it can be interpreted as the generating function for the multiplicities in the particular sector of BPS states in the SUSY gauge theories [17].

We will be interested in the critical behavior of the "area+length+corner" type statistics in the ensemble of torus knots and focus on the particular series of  $T_{n,n+1}$  uncolored knots parameterized by one integer,  $n$ . The main object, as before, is the generating function for superpolynomials,  $Z(s, a, q, t)$ , in the ensemble of  $T_{n,n+1}$  knots, where the fugacity,  $s$ , is conjugated to the index  $n$ , which defines the winding around the cycle on the solid torus (see the Fig.1),

$$Z(s, a, q, t) = \sum_{n=0}^{\infty} P_{n,n+1}(a, q, t) s^n \quad (16)$$

Hopefully there is an explicit expression for the superpolynomial of the  $T_{n,n+1}$  torus knots obtained in two different ways. The first one deals with the combinatorics of the Young diagrams and can be related to the Brownian excursion approach [18], while the second approach has been developed in [4] via the localization on the fixed points of the torus action in the moduli space of  $n$  points in  $\mathbb{C}^2$ . The second approach will be used in the next Section to compare the generating function for superpolynomials of  $T_{n,n+1}$  uncolored torus knots with the instanton contribution to the particular observable in 5D SQED with the Chern–Simons term.

We focus on the BE representation and provide a dictionary identifying the fugacities at BE side with the ones at the knot side. This dictionary establishes the translation of the BE language to the language of torus knot superpolynomials. The construction of torus knot invariants involves two more statistics for Dyck paths, the statistics of "up-corners", where the path changes direction from "up" ( $\uparrow$ ) to "right" ( $\rightarrow$ ), and the statistics of "dinv", a definition for which can be found in [18]. The superpolynomial for  $T_{n,n+1}$  knots expressed in terms of Dyck paths can be presented by the following partition function:

$$Z_n(q, t, a) = \sum_{\pi_n \in \text{Dyck paths}} q^A t^{\# \text{dinv}} a^{\# \text{corners}}, \quad (17)$$

where  $q$ ,  $a$  and  $t$  are the fugacities of area, corners and dinv correspondingly. The main object of our study is the generating function,

$$Z(q, t, a, s) = \sum_{n=0}^{\infty} Z_n(q, t, a) s^n \quad (18)$$

where  $s$  is fugacity for the length of the path conjugated to the integer  $n$ , which weights knot type (at a knot side) and Dyck path length (at a BE side).

As before, the area,  $\text{dinv}$ , and corner statistics in the ensemble of lattice paths are represented by the corresponding fugacities in the grand ensemble. To use the known critical behavior for the generating function of superpolynomials in the ensemble of torus knots, we have to consider the reduction of paths statistics to the "length+area+corners" one, switching off the "dinv" fugacity (i.e. setting  $t = 1$  in (17)).

Due to  $q \leftrightarrow t$  duality, one can equivalently consider the "length+area+corners+bounce" statistics, switching off the "bounce" fugacity. This duality can be seen, for instance, in the lowest row in the expansion in the  $a$  variable of the superpolynomial:

$$P_{n,n+1}(a, q, t) = \sum_k a^k P_{n,n+1}^k(q, t). \quad (19)$$

It turns out that the corresponding term at  $a = 0$  in the expansion coincides with the  $(q, t)$ -Catalan number, namely,  $P_{n,n+1}^0(q, t) = C_n(q, t)$ , when the duality is well-known.

In the next Section we explain the relation between the generating function for the torus knot superpolynomials and the some observables in the 5D SQED upon the particular identification of parameters. In the Section 6 we shall use the generating function for  $q$ -Narayanas to analyze the critical behavior in the torus knots ensembles classified by the HOMFLY knot invariants.

#### IV. TOWARDS THE CRITICAL BEHAVIOR IN 5D SQED

Consider now the Nekrasov-like partition function [19] in the Abelian 5D SUSY gauge theory with the massless hypermultiplet in fundamental representation and massive multiplet in the antifundamental representation in the  $\Omega$ -background. The coefficient in front of the Chern-Simons term is fixed,  $k = 1$ , the coupling constant in 5D theory is dimensionful, and the fifth coordinate is compact. The Nekrasov partition function is trivial in this theory, however we shall be interested in the vacuum matrix element  $\langle O \rangle$  of the particular operator  $O$ . Its evaluation involves the weighted sum of the integrals over the instanton moduli space, where the parameters of the  $\Omega$ -background,  $\epsilon_1, \epsilon_2$ , serve as the equivariant parameters of two torus actions for the integration over the moduli space,  $M_n$ , of  $n$  point-like instantons, all located at the origin. The instanton number is weighted with the counting parameter  $Q = e^{2\pi i \tau}$ , where  $\tau = 4\pi i \beta g^{-2}$  and  $\beta$  is the radius of the compact fifth dimension.

The desired operator  $O$  can be identified as follows [50]. First, we have to recognize the deformed Catalan

numbers,  $C_n(q, t)$ , with  $\langle O \rangle$  in the  $n$ -instanton sector. To this aim we use the following important result [11]

$$\chi^T(\text{Hilb}^n(\mathbb{C}^2, 0), V \otimes \Lambda^n V) = C_n(q, t), \quad (20)$$

which interprets the  $(q, t)$ -deformed Catalans as equivariant integrals over the moduli space of  $n$ -Abelian instantons valued in the  $n$ th power of the  $n$ -dimensional tautological bundle,  $V$ .

Looking at the representation of the  $(q, t)$ -Catalans in terms of the paths on the Young tableau, the desired operator, up to the normalization, can be written in the  $n$ -instanton sector as

$$\langle O \rangle_n \propto \left\langle \tilde{Q} Q (\text{Tr} e^\Phi) \right\rangle_n, \quad (21)$$

where  $\tilde{Q}, Q$  is the hypermultiplet,  $\text{Tr}$  substitutes the integral over the  $\mathbb{C}^2$  in the  $\Omega$ -background, and  $\Phi$  is the "long scalar" in the  $\Omega$ -background [19]. Hence, the composite operator under consideration, is the product of local and nonlocal 4-observables.

It is more convenient to use the equivalent, however a bit more symmetric formulation of the desired observable in 5D SQED [35]. Consider the 5D QED with two flavors in the fundamental representation with masses  $m, M$ , and the flavor in the antifundamental representation with a mass  $m_a$ . The following relation between the second derivative of the Nekrasov partition function and the generating function for the  $T_{n,n+1}$  knot superpolynomials, holds:

$$\begin{aligned} & \frac{e^{\beta M}}{(1+a)\beta^2} \frac{d^2 Z_{nek}(q, t, m, M, m_a, Q)}{dM dm} \Big|_{m \rightarrow 0, M \rightarrow \infty} \\ &= \sum_n Q^n (tq)^{n/2} P_{n,n+1}(q, t, a) \end{aligned} \quad (22)$$

This is equivalent to the evaluation of the correlator of local and nonlocal operators defined above due to the relation

$$\frac{e^{\beta M}}{\beta} \frac{\partial Z_{nek}}{\partial M} \Big|_{M \rightarrow \infty} = \langle (\text{Tr} e^\Phi) \rangle, \quad (23)$$

We keep the parameter  $a$  in the superpolynomial arbitrary, hence the complete list of the identifications reads as follows

$$a = -e^{-m_a \beta}, \quad t = e^{-\beta \epsilon_1}, \quad q = e^{-\beta \epsilon_2} \quad (24)$$

Thus, as above, we have the generating function depending on four fugacities.

The relation between the derivative of the Nekrasov partition function and torus knots superpolynomials is important per se, however in this paper we are focusing at the critical behavior in the topological ensembles. Therefore, what we need is i) the identification of the parameters of the gauge theory as the fugacities of the random walk, and ii) expression of the generating function

for the  $q$ -Narayana numbers. To make use of the known expression for  $q$ -Narayana's, we have to switch off one equivariant parameter,  $q = t^{-1}$ . The critical behavior is now the  $(Q, m_a, q)$ -phase space, and in this formulation it is clear that we are dealing with the 3rd order phase transition. However, the new point is that the 3rd derivative of the Nekrasov partition function is taken with respect to three distinct variables  $(m, M, Q)$  (or, equivalently, to  $(m, M, m_a)$ ). The physical interpretation of the critical behavior is given in the next Section.

## V. ON THE PHYSICAL INTERPRETATION OF CRITICAL BEHAVIOR

In this Section we discuss the physical interpretation of the critical behavior found above. As we have already argued, the critical behavior is exact due to the explicit expression of the generating function for  $q$ -Narayana's, which depends on three fugacities (chemical potentials) controlling length of the path, area under the path and corners. The interpretations of the fugacities and of the critical behavior are different at Brownian excursion side, knot side, and 5D SQED side, so we consider them separately. However, these seemingly different critical behaviors reflect one and the same generic pattern of the phase transition.

### A. Critical behavior at the Brownian excursion side

In terms of the (1+1)D random walks the critical behavior in the  $(s, a, q)$  space has the following interpretation. In the limit  $q \rightarrow 1$  the fugacity of the area is switched off and we deal with the generating function  $F(a, s)$  (see (12)) defined in the  $(s, a)$ -plane. At large finite lengths,  $n$ , the average number of corners,  $\langle C(n) \rangle$  diverges with the length,  $n$ , as  $\langle C(n) \rangle|_{n \gg 1} = \frac{n}{2}$ . The corresponding contour plot of the generating function  $F(s, a)$  is shown in the Fig.5 in the  $q = 1$ -panel. The critical lines divide the  $(s, a)$ -plane in three domains. In the first domain ("phase 1") since  $s$  is small, one has short and (in average) sufficiently wrinkled paths, with varying number of corners controlled by the fugacity  $a$ . In the second (intermediate) domain the trajectories are long since they exceed the critical value for the fugacity, again with varying number of corners, however since the generating function  $F(s, s)$  diverges, currently we have not any physical interpretation of this phase. In the third domain ("phase 2") the trajectories are long and essentially wrinkled, since for any  $s$  the value of  $a$  is bounded from below:  $a > a_{\min} = \frac{1+\sqrt{s}}{s}$ .

For  $q \neq 1$  the behavior of the system becomes much more rich and the ensemble of area- and corner-weighted Dyck paths exhibits the cascade of Airy-type (3rd order)

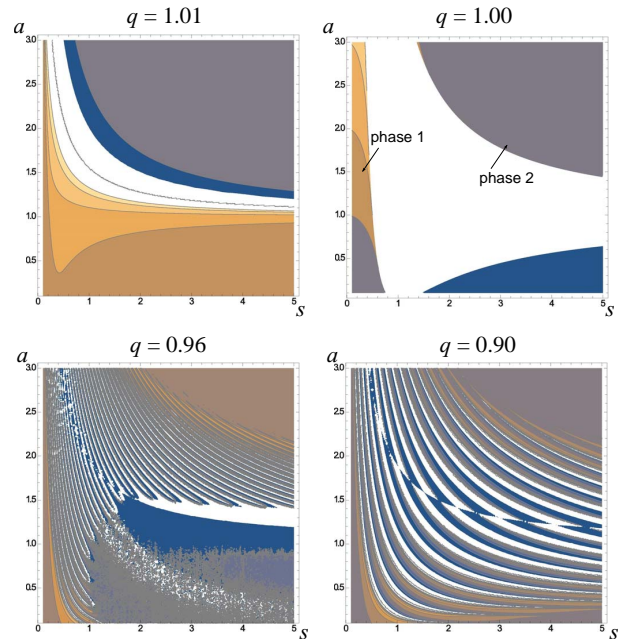


Figure 5: Relief of the generating function  $F(a, s)$  above the  $(s, a)$ -plane for different fixed values of the area fugacity,  $q$ . For  $q = 1$  the phase 1 corresponds to short paths, and the phase 2 – to long and rather wrinkled paths. For  $q \neq 1$  one sees in the  $(s, a)$ -plane the cascade of transitions with the Airy-type asymptotics.

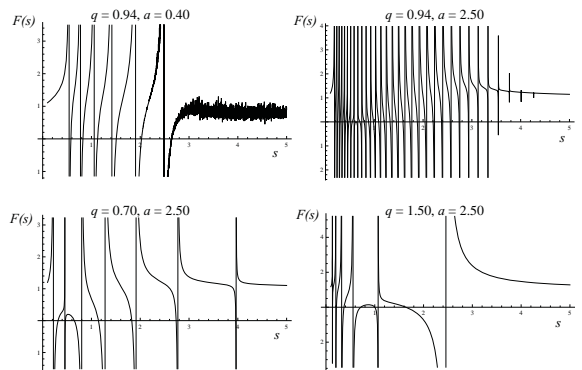


Figure 6: Plots of the generating function  $F(s)$  for few fixed values of  $q$  and  $a$ .

phase transitions as it is seen from the contour plots for the function  $F(q, s, a)$  drawn at few fixed values of  $q$  ( $q = 0.9, 0.96, 1.01$ ) – see the corresponding panels in the Fig.5.

The asymptotic behavior of the function  $F(q, a, s)$  as a function of  $s$  for fixed values  $q$  and  $a$  near the singularities (transition points) is better seen in the figure Fig.6. One clearly distinguishes the finite cascade of phase transitions at different  $q \neq 1$  and  $a$ .

For completeness, we provide in the Fig.7 two contour plots of the function  $F(q, s)$  at two fixed values of corner



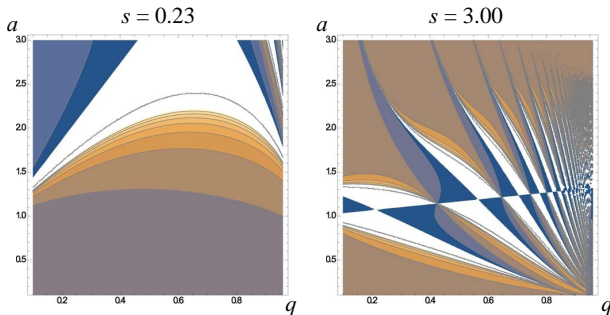


Figure 7: Plots of the generating function  $F(q, s)$  above the  $(q, s)$  plane for fixed values  $a = 0.23, 3.00$ . The cascade of transitions is clearly seen.

fugacity,  $a = 0.23$  and  $a = 3.00$ , which correspond to weakly and highly wrinkled paths.

The nature of existence of the cascade of transitions lies in the lattice origin of the problem and deals with the incommensurability of "optimal" trajectories with respect to the lattice spacing for particular values of  $a$  and  $q$  and  $s$ . Actually, it is very easy to imagine that there are conflicts between wrinkled (with large number of corners) and "inflated" (with extended area) trajectories at some specific length. On the lattice this competition leads to a finite number of phase transitions, manifested in the divergence of the generating function  $F(q, a, s)$ .

### B. Critical behavior at knot side

In the ensemble of the  $T_{n,n+1}$  uncolored torus knots, the situation when we can use the exact answer for the  $q$ -Narayana's corresponds to the unrefined case with 3D space of parameters  $(s, a, q)$ . Hence, we consider generating function for HOMFLY polynomials,  $H_n(a, q)$ , for  $T_{n,n+1}$  torus knots.

What could be the interpretation of the phase transitions in the ensembles of the torus knots? Naively, different behavior of the generating function at different ridges of the transition point, means different ability to recognize a particular knot in the ensemble. Speaking more formally, we could try to distinguish phases via the derivatives of the generating function with respect to the particular generating parameters, that is the average values of the corresponding "topological numbers". The derivative with respect to the variables yields the average number of winding, or in other terms, the type of the torus knot. However the interpretation of other topological numbers is less evident. On the other hand, these quantum numbers are quite transparent if we remember that knot invariants are related to the enumeration of BPS states. Along this approach, other generating parameters count the number of the corresponding branes.

Let us remind, that in a pure "knot framework", one

can obtain the  $SU(n)$  invariants of the knots upon the substitution

$$a = q^n \quad (25)$$

Hence, the  $q \rightarrow 1$  limit corresponds to the situation when all  $SU(n)$  invariants are "degenerate" and cannot be distinguished. At the critical line in the  $(a, s)$  plane in the ensemble of torus knots, the average winding  $\langle n \rangle$  diverges and the average value  $\langle \frac{dH}{da} \rangle$  diverges as well. On the critical line they are related as

$$\langle n \rangle \propto \left\langle \frac{dH}{da} \right\rangle \quad (26)$$

Hence, in the ensemble of torus knots, the stable limit ( $n \rightarrow \infty$ ) is reached at the critical line, where we also forget about the "nonabelian character" of knot invariants.

Note that the counting problem can be formulated in terms of the Seifert surface, which for the  $T_{n,n+1}$  torus knots reads as

$$x^n = y^{n+1} \quad (27)$$

where  $x, y \in C$ . For instance, the Alexander polynomials can be formulated in terms of the monodromy in the cohomologies related to the Seifert surface. Also, the Alexander polynomials can be interpreted in terms of the stratification of the algebra of functions on the Seifert surface. Hence, the critical behavior can be formulated in terms of the Seifert surface as well. Whether one can relate the type of the Seifert surface with the singularity type near the transition point of the generating function, is still an open question.

Let us comment on the interpretation of the cascades of the phase transitions at "torus knot side". Assuming that there are infinite number of phases in the generic situation, it is naturally to conjecture the domination of the different values or windings in different phases.

### C. Critical behavior at instanton side

Turn finally to the third part of our construction. In the ensemble of instantons, the account of three fugacities  $(s, a, q)$  corresponds to the account of the coupling constant, mass and parameter of the  $\Omega$ -deformation in the unrefined limit. The critical cascades in this case can be derived again from the analytic structure of the  $q$ -Narayana generating function,  $F(q, s, a)$ .

The limit  $q \rightarrow 1$  corresponds to switching off the  $\Omega$ -deformation. Hence, the  $(s, a)$ -phase space corresponds to the space of the mass,  $m_a$ , of the hypermultiplet in the antifundamental representation [35], and the instanton charge counting parameter,  $Q$ . The critical line in the physical  $(s, a)$ -variables is defined by the equation

$$(1 + Q - Qe^{m_a\beta})^2 = 4Q \quad (28)$$

There are two critical lines,  $Q = \left(\frac{1-e^{m_a/2}}{1-e^{m_a}}\right)^2$  and  $Q = \left(\frac{1+e^{m_a/2}}{1-e^{m_a}}\right)^2$ , defined by (28) at the particular relation between the gauge coupling and mass of the antifundamental representation. Configurations with different number of instantons dominate in the different regions of the  $(s, a)$ -space. At the critical line we have the following scaling behavior

$$\frac{d^2 Z}{dM dm_a} \propto \frac{d^2 Z}{dM dQ} \quad (29)$$

When the parameter of the  $\Omega$ -deformation is switched back again, the cascade of the singular surfaces emerges, dividing the three-dimensional parameter space in the regions. Since the parameter  $q$  corresponds in the 5D SQED to the fugacity for the angular momentum in some plane in  $C^2$ , the critical surfaces separate the regions with the different responses of our observable to the rotation. When  $a = 0$ , the antifundamental becomes massless and we have the critical cascades in  $(s, q)$ -plane; their scaling is dictated by the analytic structure of the  $q$ -Airy functions.

In the simplest case when only the  $s$ -parameter matters, the critical behavior of the  $q$ -Catalans corresponds to the following values of parameters in the 5D gauge theory

$$\epsilon_1 = 0, \quad \epsilon_2 \beta \rightarrow 0, \quad \ln s = -8\pi^2 \beta g^{-2} = 4\pi \log 2 \quad (30)$$

The general issues concerning the 3rd order phase transitions will be discussed in the separate publication [31].

## VI. AREA-WEIGHTED DYCK PATHS AND TODA SYSTEM

Let us point out that the scaling function  $C(z)$ , which appeared many times throughout the text (see, for instance, (9), or (B2)), plays also a very important role, connecting BE to the integrable systems. It is known (see [23–25] and the references therein) that  $w(z)$ , defined in (B2), is itself a generating function:

$$w(z) \Big|_{z \rightarrow \infty} \sim \sum_{k=0}^{\infty} \frac{(-1)^k}{2^k k!} \Omega_k z^{-(3k-1)/2}, \quad (31)$$

where the coefficients  $\Omega_k$  have well-defined physical sense: representing  $\Omega_k$  in the form

$$\Omega_k = 2^{(3k-1)/2} \Gamma((3k-1)/2) \langle \mathcal{B}^k \rangle$$

where  $\Gamma(\dots)$  is the gamma-function. One can show [25, 26] that  $\langle \mathcal{B}^n \rangle$  is the  $n$ th moment of the area under the Brownian excursion on the unit interval. Defining  $K_n = \frac{\Omega_n}{2^{n+1}n!}$  one sees that  $K_n$  satisfy the recursion

$$K_n = \frac{3n-4}{4} K_{n-1} + \sum_{k=1}^{n-1} K_k K_{n-k}; \quad K_0 = -\frac{1}{2}. \quad (32)$$

On the other hand, the function  $w(x)$  appears in the theory of integrable systems as the solution of the rational Painlevé II equation ( $w(x) \equiv u(x)$ ) at  $\alpha = 0$ :

$$u''(x) = 2u^3(x) + 4xu(x) + 4 \left( \alpha + \frac{1}{2} \right). \quad (33)$$

The connection of area-weighted generating function  $w(x)$  with the rational solutions of Painlevé II is not restricted exclusively by (33), and can be pushed for any  $\alpha = N + \frac{1}{2}$ , where  $N = 0, 1, 2, \dots$ . Take into account that the rational solutions of (33) can be written (see [27, 28]) as  $u(x) = -\ln \frac{\sigma_{N+1}}{\sigma_N}$ , where  $\sigma_N \equiv \sigma_N(x)$  is the  $\tau$ -function of the Toda system (see [29]), written as a Hankel determinant

$$\sigma_N = \det \begin{pmatrix} a_0 & a_1 & \cdots & a_{N-1} \\ a_1 & a_2 & \cdots & a_N \\ \vdots & \vdots & \ddots & \vdots \\ a_{N-1} & a_N & \cdots & a_{2N-2} \end{pmatrix} \quad (34)$$

and the entries  $a_n \equiv a_n(x)$  satisfy the recursion [28]

$$a_n = 2(n-2)a_{n-3} + \sum_{k=0}^{n-2} a_k a_{n-k-2}, \quad (35)$$

with  $a_0 = x$ ,  $a_1 = 1$ ,  $a_2 = x^2$ . The associated generating function,

$$G(x, s) = \sum_{j=0}^{\infty} a_j(x) (-2s)^{-j}, \quad (36)$$

obeys the Riccati equation [28] (compare to (B1))

$$-2G + G^2 + \partial_s G - (4s^2 + s^{-1})G + 4xs^2 = 0, \quad (37)$$

whose solution is

$$G(x, s) = 2s^2 + \frac{d}{ds} \ln \text{Ai}(s^2 - x). \quad (38)$$

The equation (35) at large  $n$  resembles (though being different in details) the recursion (32) for the function  $K_n$ . The connection between (32) and (35) can be set by comparing (31) and (36). Finally, we get

$$a_j = (-2)^j \sum_{k=0}^{\infty} \frac{(-1)^k \Omega_k}{2^{k-1} k!} \sum_{j=0}^{\infty} \left( \frac{1-3k}{2} \right) (-x)^m \delta_{3k+2m+1, j}, \quad (39)$$

where  $\delta_{i,j}$  is the Kronecker  $\delta$ -function. Thus, we explicitly see how the linear combinations of moments of area-weighted Brownian excursions,  $\Omega_k$ , are connected to the coefficients  $a_j$  in the expansion of the Toda  $\tau$ -function.

Let us also point out the striking similarity of the recursion equation (32) for different momenta of area-weighted Dyck paths with the summation over genus,  $g$ , the partition function of the  $U(n)_k \times U(n)_{-k}$  Chern–Simons–matter theory, also known as the ABJM theory [30]. We plan to discuss this question in details in the forthcoming paper [32].



## VII. CONCLUSION

In this paper, the consideration of the directed random walks with the "area+length+corners" statistics allows us to find a new critical behavior in the ensemble of the  $T_{n,n+1}$  torus knots and in the ensemble of instantons in 5D SQED. This critical behavior takes place in the space of fugacities and the corresponding parameters in the gauge theory. The result is exact due to the explicit answer of the generating function for paths with these statistics.

In the space of fugacities there are patterns of critical lines, and cascades of the phase transitions. The physics of the phase transitions at this critical line for the ensembles of torus knots and instantons is very rich. In both cases at the critical line the contribution to the quantity under consideration with the infinitely large topological number starts to dominate. These quantities are windings in the knot case, or topological charges in the 5D SQED. In the gauge theory we have the critical lines in the (gauge coupling, mass) parameter plane. The phase transition in this plane is physically meaningful and certainly deserved detailed investigation.

Even more profound behavior is seen when all three fugacities are nontrivial. The critical cascades are quite unexpected and physically they take place in the unrefined limit of the  $\Omega$ -deformed gauge theory. At the knot side, the critical behavior occurs in the generating function of the  $T_{n,n+1}$  knots characterized by the HOMFLY polynomials.

The following general comment is in order. One has to distinguish two "critical phenomena": phase transition and the wall-crossing phenomena. The latter is typical for the supersymmetric gauge theories when the spectrum of BPS states jumps at some curves and surfaces in the moduli space. Upon the SUSY breaking, the curves of marginal stability become the curves of phase transitions. Hence literally the wall-crossing should not be considered as a phase transition, since only the spectrum of the stable BPS states gets rearranged. However, there is a pattern of real phase transitions, even in SUSY theory, with the Argyres–Douglas superconformal point in SQCD as the notable example. In this case the particular condensates serve as the order parameter.

Do we deal with the phase transition or with the wall-crossing phenomena in our case? Formally, the 3rd derivative of the Nekrasov partition function in 5D SQED diverges, hence we could speak about the 3rd order phase transition in the instanton ensemble. Moreover, the UV properties of the condensate play the key role, and critical lines involve the hypermultiplet masses and the gauge coupling. Hence, we could speak on some similarity with the Argyres–Douglas scenario. On the other hand, at the torus knot side, we could remind the interpretation of the knot superpolynomial as the index of the BPS states in

some theory. Hence the jump in the index is better interpreted in terms of wall-crossing. As far as we know, there is no evident notion of the wall-crossing in terms of random walks, however, we could think about it in terms of the Stokes lines in quantum mechanics. Summarizing, we think that the critical behavior we have described, admits the interpretation both as the wall-crossing, and as the phase transition, depending on the viewpoint.

The critical behavior we have considered in 5D SQED is a nontrivial example of the 3rd order phase transition. The familiar example of the 3rd order phase transition in the gauge theories is the phase transition of Douglas–Kazakov type [22] in 2D pure Yang–Mills theory on the sphere characterized by the singularity (9). Somewhat similar phase transition of Douglas–Kazakov type has been observed in [43]. In all these situations the phase transitions are driven by instantons. On the other hand, our study, as well as [35], yield the clear relation between the torus knots and instantons. This suggests more general viewpoint that there is a knotting process behind the 3rd order phase transitions in the general case and the knot invariants play the role similar to the central charges of the Virasoro algebra in the 2nd order phase transitions. We shall provide the general discussion of the 3rd order phase transition in the gauge theories, knot ensembles and statistical mechanics elsewhere [31].

There are many challenging issues, related to our work, which deserve further study. For example, it would be interesting to extend the analysis to the whole ensemble of  $T_{n,m}$  torus knots and the whole set of fugacities. Also, it would be important to recognize the counterparts of the "bounces" and "corners" in the particle path integral in the continuum (i.e. off lattice) and obtain the interpretation of the external magnetic field in the BE approach as a kind of the Berry curvature from a 4D viewpoint. It would be also very interesting to extend our consideration and to relate the spectrum of the full (i.e. non-chiral) Hofstadter model at BE side with knot invariants and 4D instantons. We plan to address all these questions in [31]. It seems also very important to recognize the critical behavior observed in this paper in the ensembles of branes [37], Hopfions [38] and "ensemble" of 3D theories classified by the torus knots [39]. Another interesting question concerns the physical interpretation of the critical behavior in terms of the extended objects involved in the torus knot description along the discussion in [35, 36]. This is the explicit realization of the  $S$ -dual magnetic approach to the evaluation of the knot invariants suggested in [40].

We are grateful to E. Gorsky, A. Milekhin and N. Nekrasov for the useful discussions. The work of A.G. and K.B. was supported in part by grants RFBR-12-02-00284 and PICS-12-02-91052. The work of K.B. was also supported by the Dynasty fellowship program and Princeton Centennial Fellowship. A.G. thanks SCGP at Stony Brook University where the part of this work has been done during the Simons Workshop on Mathematics and Physics 2014 for the hospitality and support.

## Appendix A: $q$ -statistics of Dyck paths in a group-theoretic setting

### 1. Area-weighted Dyck paths as random walks on a $q$ -deformed locally-free semigroup

Let us associate each entrance of a Dyck path at a height  $x$  with the application of a "generator"  $\hat{g}_x$ . Define the semigroup  $F^+$  with the infinite set of generators  $\{\hat{g}_0, \hat{g}_1, \hat{g}_2, \dots\}$ , obeying the commutation relations

$$\hat{g}_j \hat{g}_k = \hat{g}_k \hat{g}_j \quad \forall |k - j| \geq 2 \quad (\text{A1})$$

Any Dyck path can be uniquely encoded by a "word" written in terms of generators  $\hat{g}_x$  ( $x = 1, 2, \dots$ ). For example, the Dyck path in the Fig.3 corresponds to a word

$$\mathcal{W} = \hat{g}_0 \hat{g}_3 \hat{g}_2 \hat{g}_1 \hat{g}_2 \hat{g}_1 \hat{g}_3 \hat{g}_3 \hat{g}_2 \hat{g}_1 \hat{g}_1 \hat{g}_0 \hat{g}_3 \hat{g}_2 \hat{g}_1 \hat{g}_1 \hat{g}_1 \hat{g}_0 \quad (\text{A2})$$

(see also the Fig.4). Define a "normal order" representation of words in the semigroup  $F^+$ , which consists in pushing the generators with smaller indices *as left as possible* if such a reordering does not violate the commutation relations [44]. Let us note that all Dyck paths written in terms of generators  $\hat{h}_x$  ( $x = 1, 2, \dots$ ) are automatically normally ordered. Thus, we can compute the partition function of all such Dyck paths via the transfer matrix approach, where the transfer matrix,  $R$ , defines which particular generator,  $\hat{g}_y$ , can stay next to the previous one,  $\hat{g}_x$ :

- (i) If  $x = 0$  then  $y = 0, 1, \dots, n$ ;
- (ii) If  $x \geq 1$  then  $y = x - 1, x, x + 1, \dots, n$ ;
- (iii) If  $x = n$  then  $y = n$

Graphical representation of the rules (i)-(ii) is shown in the Fig.8.

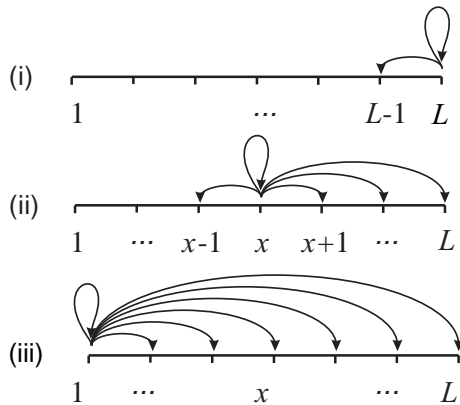


Figure 8: (a) The set of possible values which can take the index  $y$  if the previous index is  $x = 0$  (i),  $x = 1, 2, \dots, L - 1$  (ii) and  $x = L$  (iii).

In order to count heights, identify a weight  $q^x$  with the generator  $\hat{g}_x$ . Then the weighted transfer matrix,  $R(q)$ , consistent with the rules (i)-(iii) has the following form

$$R(q) = \begin{pmatrix} 1 & q & q^2 & q^3 & \dots \\ 1 & q & q^2 & q^3 & \dots \\ 0 & q & q^2 & q^3 & \dots \\ 0 & 0 & q^2 & q^3 & \dots \\ \vdots & \vdots & & & \ddots \end{pmatrix} \quad (\text{A3})$$

Define  $\mathbf{W}_n(q) = (W_n(0; q), W_n(1; q), W_n(2; q), \dots)^\top$ . The number of Dyck paths, starting at the height  $x = 0$  and returning to the height  $x = 0$  is given by the element  $W_n(0; q)$  of the vector  $\mathbf{W}_n(q)$ , where

$$W_n(0; q) = \mathbf{V} R^n(q) \mathbf{V}^\top; \quad \mathbf{V} = (1, 0, 0, \dots) \quad (\text{A4})$$

One can straightforwardly check that

$$W_n(0; q) = C_n(q) = Z_{2n}(1; q) \quad (\text{A5})$$

where  $C_n(q)$  is the  $n^{\text{th}}$  Carlitz–Riordan  $q$ -Catalan number and  $Z_n(1; q)$  is given by (4). Thus, the magnetic random walks can be equivalently described either by the transfer matrix  $T(q)$  (Eq.(3)), or by the transfer matrix  $R(q)$  (Eq.(A3)). The last description can be straightforwardly generalized to account for corners.

### 2. Area- and corner-weighted Dyck paths and $(q, a)$ -deformed locally free semigroup

The approach developed in the Section A 1 can be easily extended on counting of simultaneously area-weighted and  $\wedge$ -corner-weighted  $2n$ -step Dyck paths shown in the Fig.4. Specifically, we are interested in calculation of the partition function,  $\mathcal{Z}_{2n}(q, a)$ , defined in (17), where the summation runs over the ensemble of  $2n$ -step Dyck paths enclosing the area  $A$ , controlled by the fugacity  $q$ , and having  $M$   $\wedge$ -corners, controlled by the fugacity  $a$  (bounces are not counted, so we set  $t = 1$  in (17)).

Note that the  $\wedge$ -corner appears in position  $k$  if and only if the following condition is fulfilled:  $h_{k+1} \leq h_k$ . In the opposite situation, i.e. for  $h_{k+1} = h_k - 1$ , the  $\wedge$ -corner is not created. In terms of the transfer matrix approach, which counts the normally ordered words in the semigroup  $F^+$ , the  $\wedge$ -corner creation condition means that for all steps which do not decrease the height we should apply the weight  $a$  (the fugacity of the corner). Thus, we get the following  $(q, a)$ -deformed transfer matrix  $R(q, a)$ , where we have multiplied all entries of matrix  $R(q)$ , except the lower sub-diagonal, by a factor  $a$ :

$$R(q, a) = \begin{pmatrix} a & aq & aq^2 & aq^3 & \dots \\ 1 & aq & aq^2 & aq^3 & \dots \\ 0 & q & aq^2 & aq^3 & \dots \\ 0 & 0 & q^2 & aq^3 & \dots \\ \vdots & \vdots & & & \ddots \end{pmatrix} \quad (\text{A6})$$

(compare to (A3)). Evaluation of the area- and corner-weighted partition function,  $\mathcal{N}_n(q, a)$  is similar to (A4), namely:

$$\mathcal{N}_n(q, a) \equiv Z_{2n}(q, a) = \mathbf{V} R^n(q, a) \mathbf{V}^\top; \quad \mathbf{V} = (1, 0, 0, \dots) \quad (\text{A7})$$

In particular, for  $\mathcal{N}_n(q = 1, r)$  we get the Narayana polynomial, which is the generating function for the Narayana numbers  $\mathcal{N}_{n,k}$ ,

$$\mathcal{N}_n(q = 1, a) = \sum_{k=1}^n \mathcal{N}_{n,k} a^k = {}_2F_1(1 - n, -n, 2, a) a; \quad (\text{A8})$$

where  ${}_2F_1(1 - n, -n, 2, a)$  is the hypergeometric function and

$$\mathcal{N}_{n,k} = \frac{1}{n} \binom{n}{k} \binom{n}{k-1} \quad (\text{A9})$$

For example, for  $n = 7$  both Eq.(A7) (in which we take  $q = 1$ ) and Eq.(A9) give

$$Z_{2n}(q = 1, a) = a + 21a^2 + 105a^3 + 175a^4 + 105a^5 + 21a^6 + a^7 \quad (\text{A10})$$

meaning that in the ensemble of  $n = 14$ -step Dyck paths has 21 different configurations with 2  $\wedge$ -corners, 105 configurations with 3  $\wedge$ -corners etc.

For any  $(q, a)$  the recursion relation for  $\mathcal{N}_n(q, a)$  reads

$$\mathcal{N}_n(q, a) = a\mathcal{N}_{n-1}(q, a) + \sum_{k=1}^{n-1} q^k \mathcal{N}_k(q, a) \mathcal{N}_{n-k-1}(q, a) \quad (\text{A11})$$

(compare to (5)). The generating function

$$F(q, a, s) = \sum_{n=0}^{\infty} s^n \mathcal{N}_n(q, a)$$

obeys the functional relation

$$F(q, a, s) = 1 + (a - 1)sF(q, a, s) + sF(q, a, s)F(sq, a, s) \quad (\text{A12})$$

(compare to (6)). These polynomials coincide with the standard  $q$ -Narayana polynomials [45]. For  $q = 1$  we reproduce the generating function for Narayana numbers [46]:

$$F(a, s) = \frac{1 - (1 - a)s - \sqrt{(1 - s + sa)^2 - 4s}}{2s} \quad (\text{A13})$$

### 3. $q$ -orthogonal polynomials for area- and corner-weighted Dyck paths

Consider the recursion relation

$$\mathbf{W}_{j+1}(q, a) = R(q, a) \mathbf{W}_j(q, a) \quad (\text{A14})$$

for the vector  $\mathbf{W}_j(q, a) = (W_j(1; q, a), W_j(2; q, a), \dots)^\top$ , where  $\mathbf{W}_0(q, a) = (1, 1, \dots)^\top$  and the transfer-matrix  $R(q, a)$  is defined in (A6). One can easily rewrite (A14) for components of the vector  $\mathbf{W}_j$ :

$$W_{j+1}(x; q, a) = q^{x-1} W_j(x-1; q, a) + a \sum_{y=1}^L q^{y-1} W_j(y; q, a); \quad 1 \leq x \leq L \quad (\text{A15})$$

For the generating function,

$$F(x; q, a, s) \equiv \bar{F}(x) = \sum_{j=0}^{\infty} W_j(x; q, a) s^j$$

we obtain from (A15)

$$s^{-1} \bar{F}(x) - s^{-1} = q^{x-2} \bar{F}(x-1) + a \sum_{y=1}^L q^{y-1} \bar{F}(y) \quad (\text{A16})$$

which can be rewritten in the local form

$$\bar{F}(x+1) = (1 + sq^{x-1}(1-a)) \bar{F}(x) - sq^{x-2} \bar{F}(x-1) \quad (\text{A17})$$

This recursion relation looks pretty much similar to the recursion relation for orthogonal polynomials associated with Rogers–Ramanujan continued fraction – see [47]:

$$\begin{aligned} U_{m+1}(z; b, c) &= z(1 + bq^m)U_m(z; b, c) \\ &\quad - cq^{m-1}U_{m-1}(z; b, c); \quad (m > 0) \\ U_0(z; b, c) &= 1; \quad U_1(z; b, c) = z(1 + b) \end{aligned} \quad (\text{A18})$$

The function  $U_m(z; c, d)$  has the following explicit expression

$$U_m(z; b, c) = \sum_{k=0}^{[m/2]} \frac{(-b; q)_{m-k} (q; q)_{m-k} z^{m-2k} (-c)^k}{(-b; q)_k (q; q)_k (q; q)_{m-2k}} q^{k(k-1)} \quad (\text{A19})$$

where  $(b; q)_k = \prod_{j=0}^{k-1} (1 - bq^j)$  is the Pochhammer symbol.

By setting

$$z = 1; \quad b = \frac{s(1-a)}{q}; \quad c = \frac{s}{q}; \quad m = x \quad (\text{A20})$$

we get the following expression for the generating function  $F(x; q, a, s)$

$$F(x; q, a, s) = U_x \left( 1; \frac{s(1-a)}{q}, \frac{s}{q} \right) \quad (\text{A21})$$

The function  $F(x; q, a, s)$  can be represented as a continued fraction of Rogers–Ramanujan [47]:

$$\begin{aligned}
 F(q, a, s) &= \lim_{x \rightarrow \infty} F(x; q, a, s) \\
 &= \frac{1 + \frac{(1-a)s}{q}}{1 + s(1-a) - \frac{s}{1 + s(1-a)q - \frac{sq}{1 + s(1-a)q^2 - \frac{sq^2}{1 - \dots}}}} \\
 &= \frac{A_q(s; s(1-a))}{A_q(s/q; s(1-a)/q)} \quad (\text{A22})
 \end{aligned}$$

where  $A_q(s, s(1-a))$  is the extension of the  $q$ -Airy function  $A_q(s)$  defined in (8). The function  $A_q(s; s(1-a))$  reads

$$A_q(s; s(1-a)) = \sum_{k=0}^{\infty} \frac{q^{k^2} (-s)^k}{(q; q)_k (-s(1-a); q)_k} \quad (\text{A23})$$

At  $a = 1$  Eq.(A23) coincides with  $A_q(s)$  in Eq.(8).

## Appendix B: On relation with the hydrodynamical equations

In this Appendix we shall mention that the asymptotics (9) describes the scaling of top line in a bunch of directed vicious walks. Proceeding as in [48], take the ensemble of  $N$  vicious walkers, define the averaged position of the top line and consider its fluctuations near the averaged position. In such a description all vicious walkers lying below the top line play a role of a "mean field", which pushes the top line to some "atypical" equilibrium position, around which it fluctuates. It is naturally to suppose that the fluctuations of the top line in a mean-field approximation have the same scaling as the fluctuations of the "inflated" Brownian excursion with fixed area under the path. One actually can show that, following the line of reasoning of the work [49]. The solution of the inviscid Burgers equation

$$\partial_t u_0(x, t) + u_0(x, t) \partial_x u_0(x, t) = 0$$

is  $u_0(x, t = N) = \frac{x}{2N} \pm \frac{\sqrt{x^2 - 4N}}{2N}$  and gives the Wigner semicircle law centered at the point  $\frac{x}{2N}$ . One can smear the function  $u_0(x, t)$  near the boundary value,  $x = x_c = \pm 2\sqrt{N}$ , adding by hands the Gaussian fluctuations, i.e. passing to the Burgers equation with a weak diffusivity ( $0 < \nu \ll 1$ ),

$$\partial_t u(x, t) + u(x, t) \partial_x u(x, t) = -\nu \partial_{xx} u(x, t)$$

Seeking for weakly fluctuating solutions of viscous Burgers equation near the top line, ( $t = N$ ), in the form [49]

$$\begin{cases} x = x_c + \nu^\alpha y = 2t^{1/2} + \nu^\alpha y; \\ u(x, t) = \frac{x_c}{2t} + \nu^\beta w(s, t) = t^{-1/2} + \nu^\beta w(y, t) \end{cases}$$

and substituting the ansatz for  $u(x, t)$  into the viscous Burgers equation, one gets the equation for  $w(y, t)$ , which for  $\alpha = 2/3$ ,  $\beta = 1/3$  and appropriate boundary conditions is transformed in the limit  $\nu \rightarrow 0$  into the dimensionless Ricatti equation [49]

$$-yt^{-3/2} + \frac{1}{2}w^2 + \partial_y w = 0, \quad (\text{B1})$$

having the solution (for  $t = N$ )

$$w(z) = 2 \frac{d}{dz} \ln \text{Ai}(2^{-1/3} z) \quad z = yN^{-1/2}. \quad (\text{B2})$$

In (B2) one can recognize the singular part of the grand partition function of the "area+length"-weighted Brownian excursion. To make this connection precise, define the partition function  $Z_n(A)$  of  $n$ -step directed 2d random walk in the upper half-plane of the square lattice (i.e. the Brownian excursion) with the fixed area,  $A$ . Thus, one can straightforwardly identify  $Z(s, q)$  with  $u(y, N) = N^{-1/2} + \nu^{1/3} w(y, N)$  under the following redefinitions:

$$\begin{cases} 1 - q \leftrightarrow \nu, \\ \frac{1}{4} - s \leftrightarrow 2^{-7/3} y N^{1/6}. \end{cases} \quad (\text{B3})$$

- 
- [1] S. Nechaev *Statistics of knots and entangled random walks* (WSPC: Singapore, 1996), [arXiv:cond-mat/9812205](#)
  - [2] L. Kauffman and H. Saleur, Free fermions and the Alexander-Conway polynomial, *Comm. Math. Phys.* **141**, 293 (1991)
  - [3] M. Rosso and V. Jones, On the invariants of torus knots derived from quantum groups, *J. Knot Theory and Ramific.* **2**, 97 (1993)
  - [4] E. Gorsky and A. Negut, Refined knot invariants and Hilbert schemes, [arXiv:1304.3328](#) [math.RT]
  - [5] M. Aganagic and S. Shakirov, Refined Chern-Simons Theory and Topological String, [arXiv:1210.2733](#) [hep-th]
  - [6] H. Ooguri and C. Vafa, Knot invariants and topological strings, *Nucl. Phys. B* **577**, 419 (2000) [[hep-th/9912123](#)]
  - [7] A. Brini, B. Eynard, and M. Marino, Torus knots and mirror symmetry, *Annales Henri Poincaré* **13**, 1873 (2012), [arXiv:1105.2012](#) [hep-th]
  - [8] L. Carlitz and J. Riordan, Two element lattice permutation numbers and their  $q$ -generalization, *Duke J. Math.* **31**, 371-388 (1964); J. F rlinger and J. Hofbauer,  $q$ -Catalan numbers, *J. Comb. Th. A* **40**, 248-264 (1985)
  - [9] J. Cigler,  $q$ -Catalan numbers and  $q$ -Narayana polynomials, [arXiv:math/0507225](#) [math.CO]
  - [10] A L Owczarek and T. Prellberg, Pressure exerted by a

- vesicle on a surface, *J. Phys. A: Math. Theor.* **47** (2014) 215001; [arXiv:1311.2174](#)
- [11] M. Haiman,  $t, q$ -Catalan numbers and the Hilbert scheme *Discrete Math.* **193**, 201-224 (1998)
- [12] J. Haglund, Conjectured statistics for the  $q, t$ -Catalan numbers, *Adv. Math.* **175** (2003) 319
- [13] S. Mashkevich and S. Ouvry, Area Distribution of Two-Dimensional Random Walks on a Square Lattice *J. Stat. Phys.* **137**, 71-78 (2009);  
S. Matveenko and S. Ouvry, The area distribution of two-dimensional random walks and non-Hermitian Hofstadter quantum mechanics, *J. Phys. A: Math. Theor.* **47** 185001 (2014)
- [14] T. Prellberg and R. Brak, Critical exponents from non-linear functional equations for partially directed cluster models, *J. Stat. Phys.*, **78**, 701-730 (1995)
- [15] C. Richard and A. J. Guttmann, and I. Jensen, Scaling function and universal amplitude combinations for self-avoiding polygons, *J. Phys. A: Math. Gen.* **34**, L495-L501 (2001)
- [16] C. Richard, Scaling Behaviour of Two-Dimensional Polygon Models, *J. Stat. Phys.*, **108**, 459-493 (2002)
- [17] N.M. Dunfield, S. Gukov and J. Rasmussen, The Superpolynomial for knot homologies, [math/0505662](#) [[math.GT](#)]
- [18] A. Oblomkov, J. Rasmussen and V. Shende with appendix of E. Gorsky, The Hilbert scheme of a plane curve singularity and the HOMFLY homology of its link, [arxiv 1201.2115](#)
- [19] N.A. Nekrasov, Seiberg-Witten prepotential from instanton counting, *Adv. Theor. Math. Phys.* **7**, 831 (2004) [[hep-th/0206161](#)].
- [20] D. Gaiotto, L. Rastelli, and S.S. Razamat, Bootstrapping the superconformal index with surface defects, *JHEP* **1301**, 022 (2013) [[arXiv:1207.3577](#) [[hep-th](#)]]
- [21] T. Banks and A. Casher, Chiral symmetry breaking in confining theories, *Nuclear Physics B* **169**, 103-125 (1980)
- [22] M.R. Douglas and V.A. Kazakov, Large  $N$  phase transition in continuum QCD<sub>2</sub>, *Physics Letters B*, **319**, 219-230 (1993)
- [23] P. Flajolet, and G. Louchard, Analytic Variations on the Airy Distribution, *Algorithmica*, **31**, 361-377 (2001)
- [24] C. Richard, Area distribution of the planar random loop boundary, *J. Phys. A: Math. Gen.*, **37**, 4493 (2004)
- [25] S. Janson, Brownian excursion area, Wrights constants in graph enumeration, and other Brownian areas, *Probability Surveys*, **4**, 80-145 (2007)
- [26] M.J. Kearney, and S.N. Majumdar, On the area under a continuous time Brownian motion till its first-passage time, *J. Phys. A: Math. Gen.* **38**, 4097 (2005)
- [27] K. Kajiwara, and Y. Ohta, Determinant structure of the rational solutions for the Painlevé II equation, *J. Math. Phys.*, **37**, 4693 (1996)
- [28] K. Iwasaki, K. Kajiwara, and T. Nakamura, Generating function associated with the rational solutions of the Painlevé II equation, *J. Phys. A: Math. Gen.*, **35**, (2002) L207-L211
- [29] K. Kajiwara, M. Mazzocco, and Y. Ohta, A remark on the Hankel determinant formula for solutions of the Toda equation *J. Phys. A: Math. Theor.*, **40**, (2007) 12661
- [30] H. Fujia, S. Hirano, and S. Moriyama, Summing Up All Genus Free Energy of ABJM Matrix Model, [arXiv:1106.4631](#) [[hep-th](#)]
- [31] K. Bulycheva, A. Gorsky, A. Milekhin and S. Nechaev, in preparation
- [32] A. Gorsky, A. Milekhin and S. Nechaev, in preparation
- [33] J. Baik, P. Deift, and K. Johansson, On the distribution of the length of the longest increasing subsequence of random permutations *J. Amer. Math. Soc.* **12**, 1119 (1999)
- [34] C.A. Tracy and H. Widom, Comm. Math. Phys. Level-spacing distributions and the Airy kernel **159**, 151 (1994)
- [35] A. Gorsky and A. Milekhin, Condensates and instanton - torus knot duality. Hidden Physics at UV scale, [arXiv:1412.8455](#) [[hep-th](#)]
- [36] A. Gorsky and A. Milekhin, in preparation
- [37] K. Bulycheva and A. Gorsky, BPS states in the Omega-background and torus knots, *JHEP* **1404**, 164 (2014) [arXiv:1310.7361](#) [[hep-th](#)]
- [38] M. Kobayashi and M. Nitta, Torus knots as Hopfions, *Phys. Lett. B* **728**, 314 (2014), [arXiv:1304.6021](#) [[hep-th](#)]
- [39] H.-J. Chung, T. Dimofte, S. Gukov and P. Sulkowski,  $3d - 3d$  Correspondence Revisited, [arXiv:1405.3663](#) [[hep-th](#)]
- [40] E. Witten, Fivebranes and Knots, [arXiv:1101.3216](#) [[hep-th](#)]  
D. Gaiotto and E. Witten, Knot Invariants from Four-Dimensional Gauge Theory, *Adv. Theor. Math. Phys.* **16**, 935 (2012) [arXiv:1106.4789](#) [[hep-th](#)]
- [41] E. Gorsky, Zeta Functions in Algebra and Geometry, *Contemp. Math.* **566**, 213-232; [arxiv 1013.0916](#)
- [42] N. Nekrasov and A. Okounkov, Seiberg-Witten theory and random partitions, [hep-th/0306238](#)  
L. Baulieu, A. Losev, and N. Nekrasov, Chern-Simons and twisted supersymmetry in various dimensions, *Nucl. Phys. B* **522**, 82 (1998) [[hep-th/9707174](#)]
- [43] A. Marshakov and N. Nekrasov, Extended Seiberg-Witten Theory and Integrable Hierarchy, *JHEP* **0701**, 104 (2007) [[hep-th/0612019](#)]
- [44] A.M. Vershik, S. Nechaev, R. Bikbov, Statistical properties of locally free groups with application to braid groups and growth of heaps, *Comm. Math. Phys.*, **212**, 469-501 (2000)
- [45] J. Cigler,  $q$ -Catalan numbers and  $q$ -Narayana polynomials, [ArXiv: math/0507225](#)
- [46] E. Deutsch, Dyck path enumeration, *Discrete Mathematics* **204**, 167-202 (1999)
- [47] W.A. Al-Salam and M. Ismail, Orthogonal polynomials associated with the Rogers-Ramanujan continued fraction, *Pacific J. Math.* **104**, 269, (1983)
- [48] P.L. Ferrari, M. Praehofer, and H. Spohn, Stochastic Growth in One Dimension and Gaussian Multi-Matrix Models, In *proceedings of the 14th International Congress on Mathematical Physics (ICMP 2003)*, World Scientific (Ed. J.-C. Zambrini) (2006), 404-411, [arXiv: math-ph/0310053](#)
- [49] J.-P. Blaizot, M.A. Nowak, Large  $N_c$  confinement, universal shocks and random matrices, Lectures given at the 49 Cracow School of Theoretical Physics, May 31 - June 10 (2009) Zakopane, Poland, [arXiv:0911.3683](#)
- [50] We are grateful to N. Nekrasov for important discussion on this point and crucial suggestion